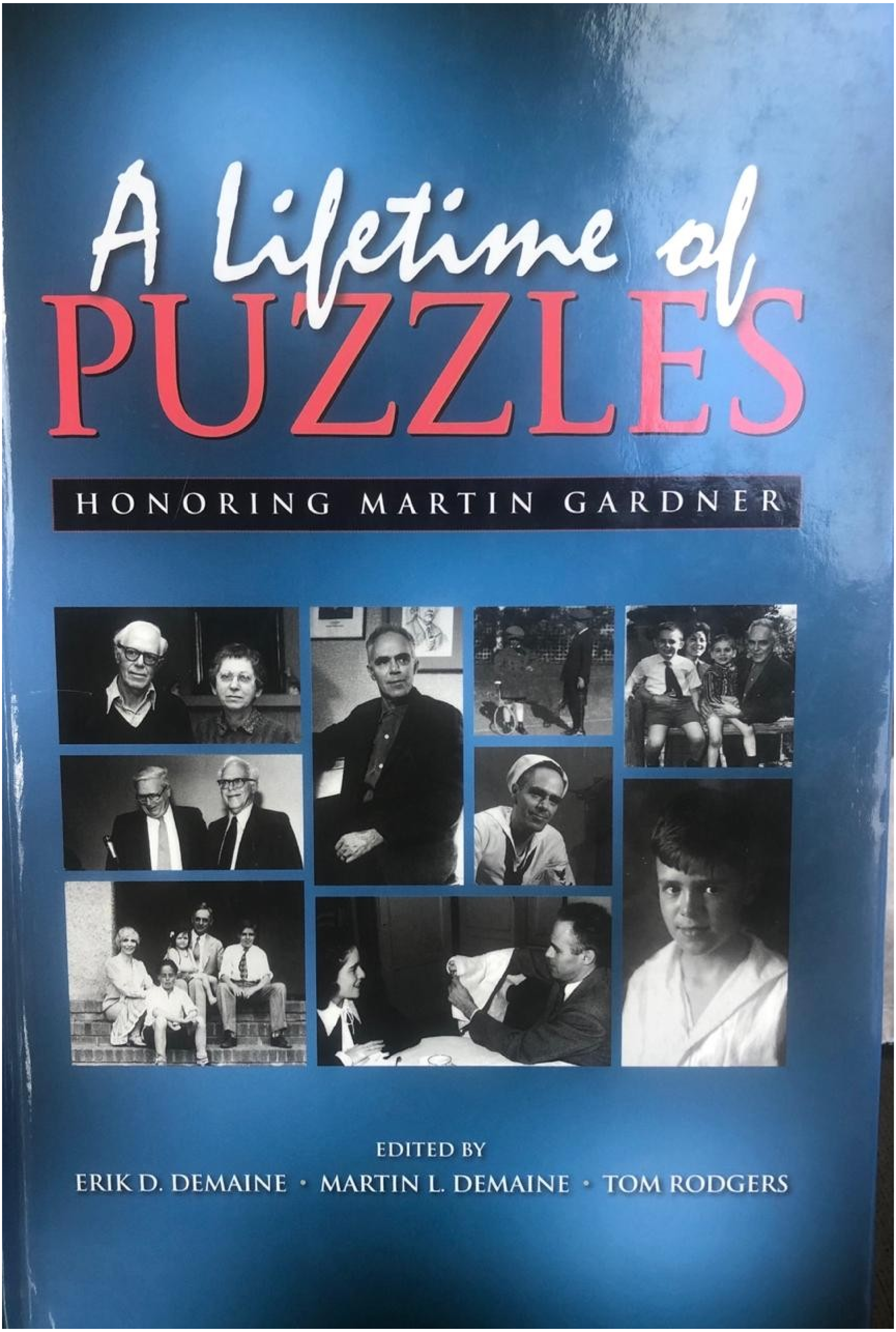

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Magic, Antimagic, and Talisman Squares

Rodolfo Kurchan

Magic squares are a classic topic in recreational mathematics. Martin Gardner discusses them in 12 of his 15 books on mathematical games; see, for example, his article "Magic Squares and Cubes" [1]. Here I describe some modern variations on magic squares and my findings about these squares by myself and others. All of my solutions were found without using computers.

Pandigital Magic Squares

A *magic square* is an $n \times n$ array of numbers such that the rows, columns, and main diagonals produce the same sum, called the *magic sum*. Here is a simple example, with a magic sum of 15:

6	1	8
7	5	3
2	9	4

A decimal number is *pandigital* if it uses precisely all ten digits and zero is not the leading digit. A *pandigital magic square* consists only of pandigital numbers and has a pandigital magic sum. In 1989, Rudolf Ondrejka [5] posed this problem: what is the pandigital magic square with the smallest pandigital magic sum?

In 1991, I found this solution [2], with pandigital magic sum 4,129,607,358:

1,037,956,284	1,036,947,285	1,027,856,394	1,026,847,395
1,026,857,394	1,027,846,395	1,036,957,284	1,037,946,285
1,036,847,295	1,037,856,294	1,026,947,385	1,027,956,384
1,027,946,385	1,026,957,384	1,037,846,295	1,036,857,294

In 2003, I found an improved solution [3], with pandigital magic sum 4,120,736,958:

1,034,728,695	1,035,628,794	1,024,739,685	1,025,639,784
1,024,639,785	1,025,739,684	1,034,628,795	1,035,728,694
1,035,629,784	1,034,729,685	1,025,638,794	1,024,738,695
1,025,738,694	1,024,638,795	1,035,729,684	1,034,629,785

In 2004, Carlos Rivera [6] quested for the smallest 3×3 pandigital magic square. By a computer search, he found the following great minimal solution, with pandigital magic sum 3,205,647,819:

1,057,834,962	1,084,263,579	1,063,549,278
1,074,263,589	1,068,549,273	1,062,834,957
1,073,549,268	1,052,834,967	1,079,263,584

Some interesting open problems about pandigital magic squares are the following:

(a) What is the pandigital magic square with the largest sum?

(b) Is there a 5×5 or larger pandigital magic square?

Antimagic Squares

An *antimagic square* is an $n \times n$ array of the numbers from 1 to n^2 such that the rows, columns, and main diagonals produce different sums, and the sums form a consecutive sequence of integers. Antimagic squares were invented by J. A. Lindon in 1962 [4, p. 103]. Here is an example:

				29
6	8	9	7	**30**
3	12	5	11	**31**
10	1	14	13	**38**
16	15	4	2	**37**
35	**36**	**32**	**33**	**34**

In 2004, I found this 5×5 antimagic square that contains in its center a 3×3 magic square [7]:

					59
7	8	24	22	2	**63**
4	16	9	14	21	**64**
25	11	13	15	5	**69**
6	12	17	10	23	**68**
18	20	3	1	19	**61**
60	**67**	**66**	**62**	**70**	**65**

In 2005, I found a 6×6 antimagic square that contains in its center a 4×4 magic square:

						108
1	36	34	33	2	3	**109**
35	26	13	12	23	6	**115**
27	15	20	21	18	5	**106**
10	19	16	17	22	30	**114**
9	14	25	24	11	29	**112**
31	7	8	4	28	32	**110**
113	**117**	**116**	**111**	**104**	**105**	**107**

An interesting open problem is to find an $n \times n$ magic square that contains in its center an $(n-2) \times (n-2)$ antimagic square, or to prove that this is impossible.

Talisman Squares

The *Talisman constant* of an $n \times n$ array of the numbers from 1 to n^2 is the minimum difference between any element and one of its eight immediate neighbors (including diagonal neighbors). A *Talisman square* is an $n \times n$ array with the largest possible Talisman constant over all $n \times n$ arrays of the numbers from 1 to n^2. Talisman squares were invented by Sidney Kravitz [4, p. 110]. As an example, consider the following two 4×4 squares:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

9	5	11	7
13	1	15	3
10	6	12	8
14	2	16	4

The left square is the well-known Dürer Magic Square (appearing in Dürer's *Melancholia I* engraving from 1514) and has a Talisman constant of 1. On the other hand, the right square has a

Talisman constant of 3. So, the left square cannot be a Talisman square, while we may assert that the right square is a Talisman square because 3 is the largest Talisman constant possible for any 4×4 square.

But how do we construct an $n \times n$ Talisman square for any given order n? Carlos Rivera and I [8] have been studying this problem, and we have found a pair of algorithms (one algorithm for even values of n, the other for odd n) that we conjecture produce a Talisman square for any given order n, as desired. Instead of providing long-winded and boring general instruction rules, we will demonstrate the algorithms using a couple of examples and some explanations about them.

Even n

We obtain a Talisman constant of $n^2/4 - 1$ for even n. Here is an example of our algorithm applied to $n = 6$, where the Talisman constant is 8:

19	10	22	13	25	16
28	1	31	4	34	7
20	11	23	14	26	17
29	2	32	5	35	8
21	12	24	15	27	18
30	3	33	6	36	9

As you may have noticed, the filling pattern in the previous example divides the numbers $1, 2, 3, ..., n^2$ into four sets (S_1, S_2, S_3, S_4), each with $n^2/4$ consecutive numbers, as follows:

$$\begin{aligned} S_1 &= \{1, \quad 2, \quad 3, \quad \ldots, \quad X-1\}, \\ S_2 &= \{X, \quad X+1, \quad X+2, \quad \ldots, \quad Y-1\}, \\ S_3 &= \{Y, \quad Y+1, \quad Y+2, \quad \ldots, \quad Z-1\}, \\ S_4 &= \{Z, \quad Z+1, \quad Z+2, \quad \ldots, \quad n^2\}. \end{aligned}$$

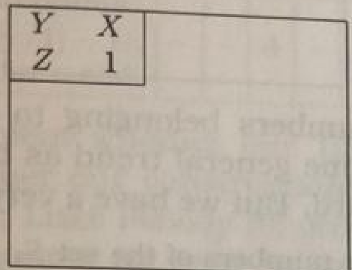
The $n^2/4$ consecutive numbers of each set are allocated in the same general trend:

> Starting from a certain specific position inside the four cells in the upper-left corner, the rest of the numbers of each set are allocated consecutively according to the rule "every two cells downward and rightward." At the moment you finish allocating the last number of the first set, you know the first number of the following set, and so on.

So, the only important thing you should know in advance is the cells in which the first numbers of each set $(1, X, Y, \text{and } Z)$ must be allocated. The answer is this:

1 goes in the cell $(2,2)$,
X goes in the cell $(1,2)$,
Y goes in the cell $(1,1)$, and
Z goes in the cell $(2,1)$,

as shown below. Moreover, if you want to know the values in advance, $X = n^2/4+1$, $Y = n^2/2+1$, and $Z = 3n^2/4+1$, but this is not really necessary to know.



We call this filling pattern "22A": "22" because it starts in the cell $(2,2)$, and "A" because the four starting numbers of each set—$1, X, Y$, and Z—describe the profile of a letter "A."

Odd n

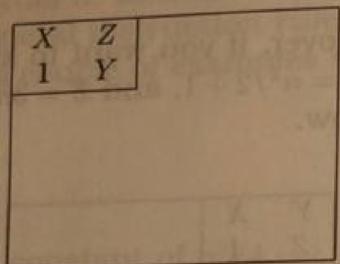
We obtain a Talisman constant of $\lfloor n(n-1)/4 \rfloor$, the integer immediately below $n(n-1)/4$, for odd n. Here is an example of our algorithm applied to $n = 7$, where the Talisman constant is 10:

13	40	17	32	21	36	25
1	29	4	44	7	47	10
14	41	18	33	22	37	26
2	30	5	45	8	48	11
15	42	19	34	23	38	27
3	31	6	46	9	49	12
16	43	20	35	24	39	28

As before, there are four sets (S_1, S_2, S_3, S_4) of consecutive numbers. Now, however, the four sets have distinct quantities of integers. Again, the starting numbers of the four sets, 1, X, Y, and Z, are allocated in the four cells in the upper-left corner, but now

1 goes in the cell $(2, 1)$,
X goes in the cell $(1, 1)$,
Y goes in the cell $(2, 2)$, and
Z goes in the cell $(1, 2)$,

as shown below. We call this pattern "21N" for analogous reasons as before.



The consecutive numbers belonging to each of the four sets are allocated in the same general trend as before: every two cells, downward and rightward. But we have a very important difference:

> When allocating the numbers of the set S_3, starting in column $4+2(\lfloor c/4 \rfloor - 1)$, shift upward by one cell all the cells that would receive the corresponding numbers for this column. The same happens with all columns rightward of this column.
>
> Consequently, when allocating the numbers of the set S_4, starting at column $4 + 2(\lfloor c/4 \rfloor - 1)$, shift downward by one cell all the cells that would receive the corresponding numbers for this column. The same happens with all columns rightward of this column.

Summary

Talisman squares are constructed as follows.

- For n even:

  Use the filling pattern 22A[1] for the starting numbers (1, X, Y, and Z) of the four sets (S_1, S_2, S_3, and S_4) of $n^2/4$ consecutive numbers; allocate the numbers of each set using the general procedure "every two cells downward, rightward." Proceeding this way, we obtain a Talisman constant of $n^2/4 - 1$.

[1] In fact, for even n, we have found two more general patterns that produce the same Talisman constant. We have selected the pattern 22A because it seems appropriate for producing Talisman rectangles, as well. However, this is still a work in progress.

- For n odd:

  Use the filling pattern 21N for the starting numbers (1, X, Y, and Z) of the four sets (S_1, S_2, S_3, and S_4) of consecutive numbers; allocate the numbers of each set using the general procedure "every two cells downward, rightward." For the sets S_3 and S_4, shift upward and downward, respectively, the starting cell in each column from $4+2(\lfloor c/4 \rfloor - 1)$ rightward. Proceeding this way, we obtain a Talisman constant of $\lfloor n(n-1)/4 \rfloor$.

n, order of Talisman square	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Talisman constant: $n^2/4 - 1$, for n even; $\lfloor n(n-1)/4 \rfloor$ for n odd.	1	3	5	8	10	15	· 18	24	27
First shifted column: $4+2(\lfloor c/4 \rfloor - 1)$, sets S_3 and S_4, just for n odd.	–	–	4	–	4	–	6	–	6

It is an open problem whether our Talisman constants can be improved, or whether our constructions are indeed Talisman squares. In May 2004, Luke Pebody [8] proved that our algorithm produces Talisman squares for even n. But, the situation seems significantly more complicated for odd n.

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