# **The Partridge Puzzle** by Robert T. Wainwright

**Last update January 8, 2024**

**https://mathpuzzle.com/partridge.html**

Looking for the smallest uncovered area in the Partridge Puzzle

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

The above statement is provable by induction.  It can also be shown with a picture.  A cube with a side of 8 can be divided into 8 squares with a side of 8.  Similarly, the cubes of size 7, 6, 5, 4, 3, 2 and 1 can be divided into stack of squares. Can these 36 squares will fit perfectly into a square with a side of 36?  Mathematically, nothing seems to prevent a solution.

Robert Wainwright presented this puzzle at the second Gathering for Gardner conference.  He gave a solution to the order 12 Partridge Puzzle, and asked if anything smaller was solvable.  He also discussed material from previous Gardner articles, including perfect squares (Mathworld Link), and optimal packings (Erich's Packing Center link).  Example perfect square: pack squares with sides (1 2 3 4 5 7 8 10 12 13 14 15 16 19 21 28 29 31 32 37 38 41 44) into a square with side 110.  Answer by A. J. W. Duijvestijn.

Several noted solvers (Bill Cutler, William Marshall, Michael Reid, Nob Yoshigahara) found solutions to the Partridge Puzzle.  Bill proved that the order-7 Partridge problem had no solution.  Bill found 2332 distinct order-8 solutions.  One of them is below.  The pieces are easy to make, and not too different in size.  It's still very hard to solve.  Can you find one of the other solutions?

More variants here: https://erich-friedman.github.io/mathmagic/0802.html?fbclid=IwAR3dlEybI1iDq06ALO8ROxgW5fdhomZ9d8SjWD-B9dBEKgvrssXWpEwQvCI

## PROBLEM 1:

$A^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 + 7x\ 7^2 + 8x\ 8^2 + 9x\ 9^2 + 10x\ 10^2 + 11x\ 11^2 \ldots Nx\ N^2$

What is the smallest area that is left uncovered?

The sequence for OEIS at the moment is: 0,4,4,16,13,8,8,0 and all solutions for n>8 seems to have solution

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| A | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| L | 0 | 4 | 4 | 16 | 13 | 8 | 8 |

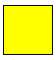
| N | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|
| A | 36 | 45 | 55 | 66 | 78 | 91 | 104 |
| L | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

N 5, N 6, N 10 and N 11 solutions by **Pontus von Brömssen**

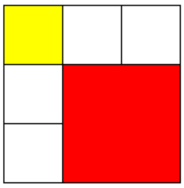
N 7 solution by **by Carl F. Schwenke**

N 8 solution by **Bill Cutler**

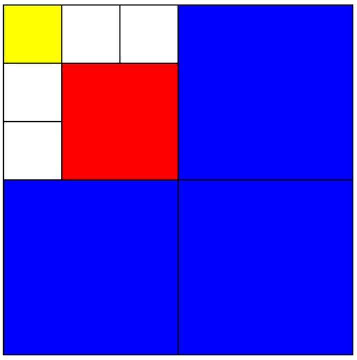
$1^2 = 1x\ 1^2 = 0$



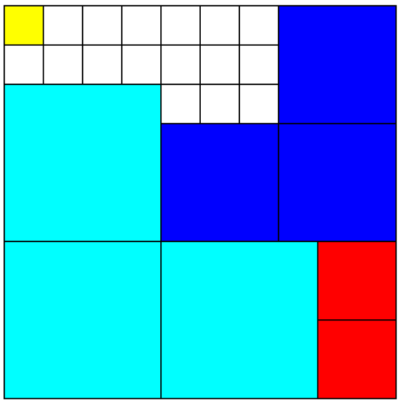
$3^2 = 1x\ 1^2 + 2x\ 2^2 = 4$



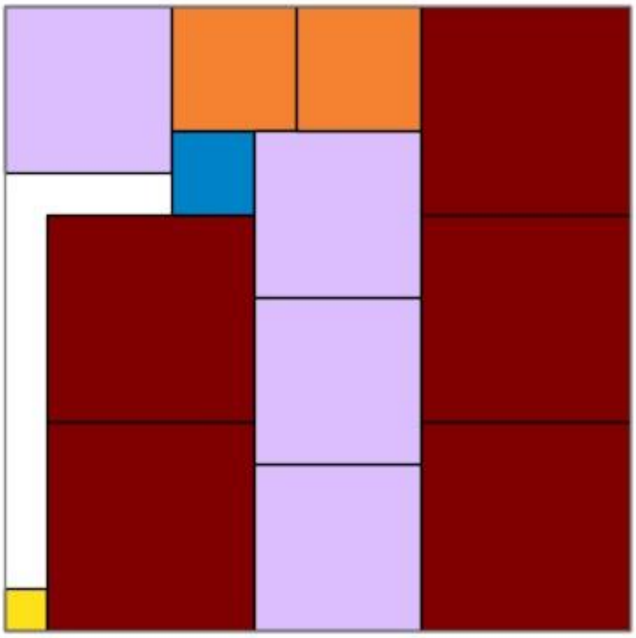
$6^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 = 4$



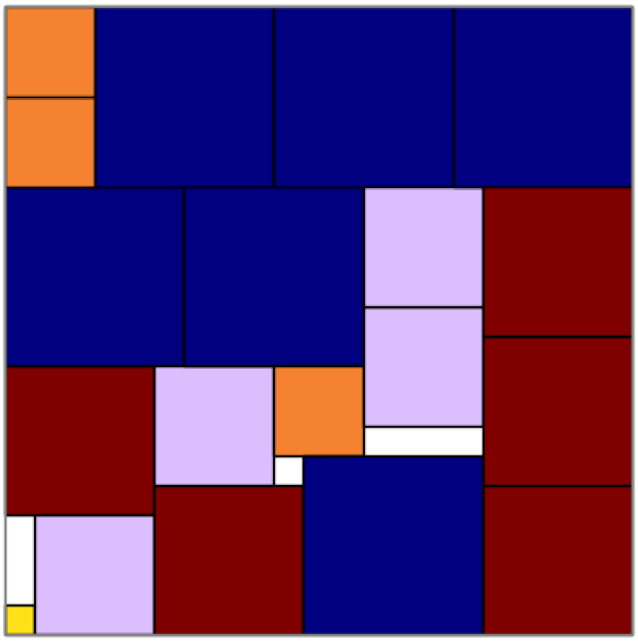
$10^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 = 16$



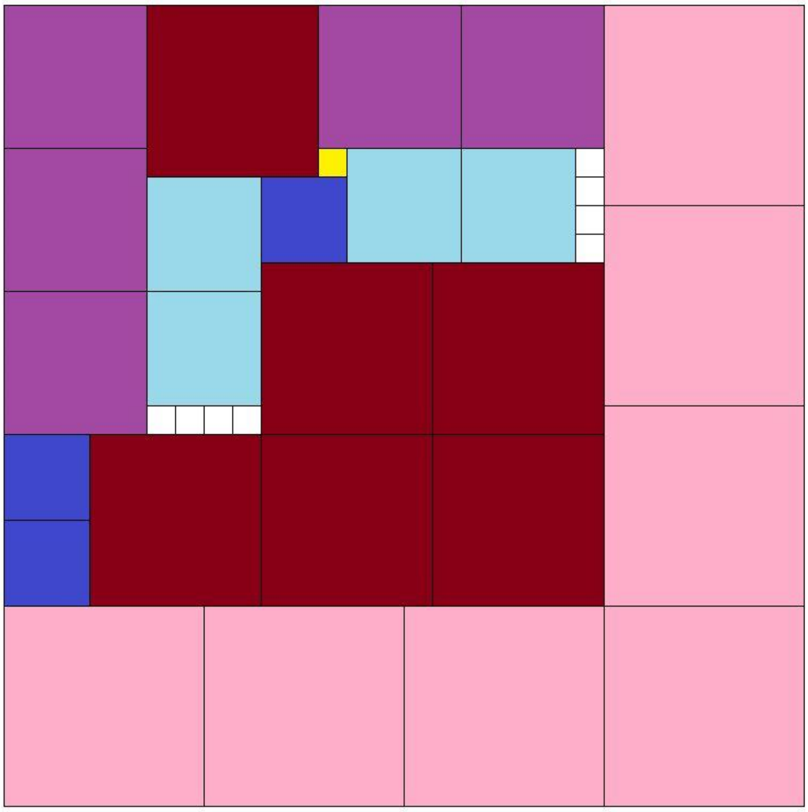
$15^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 = 13$ by Pontus von Brömssen



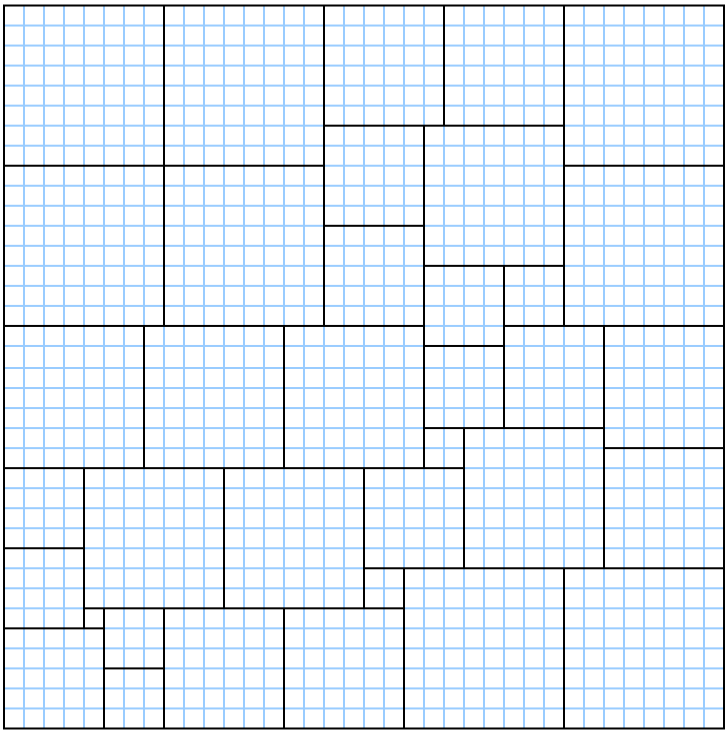
$21^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 = 8$ by Pontus von Brömssen



$28^2$ = 1x $1^2$ + 2x $2^2$ +3x $3^2$ + 4x $4^2$ + 5x $5^2$ + 6x $6^2$ + 7x $7^2$ = 8 by Carl F. Schwenke



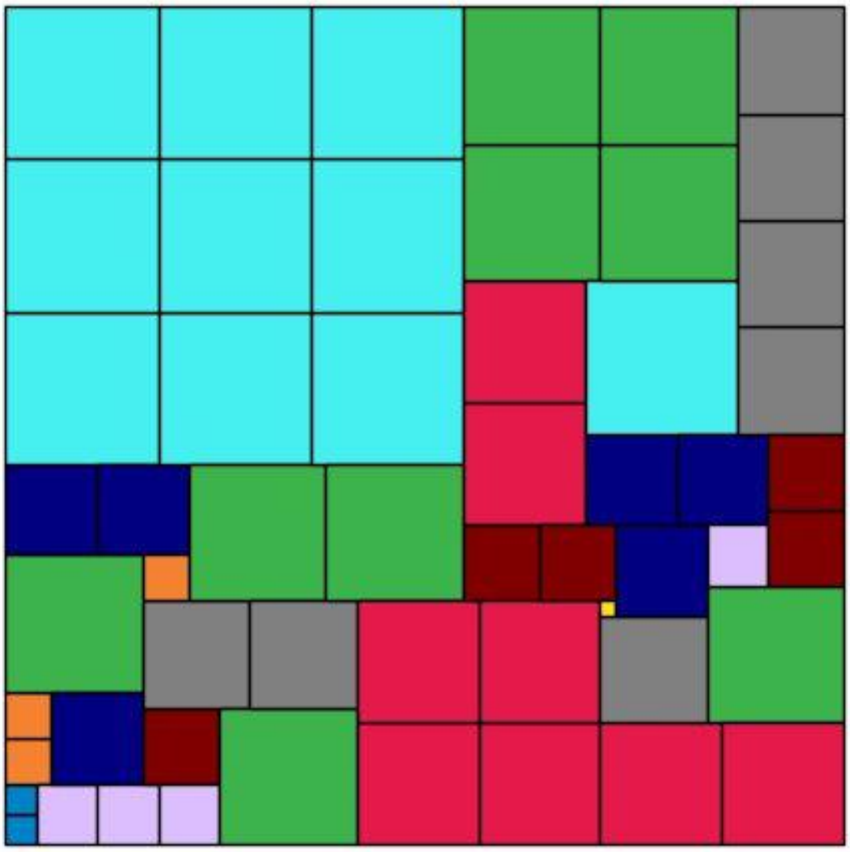
$36^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 + 7x\ 7^2 + 8x\ 8^2 = 0$ by Bill Cutler



$45^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 + 7x\ 7^2 + 8x\ 8^2 + 9x\ 9^2 = 0$

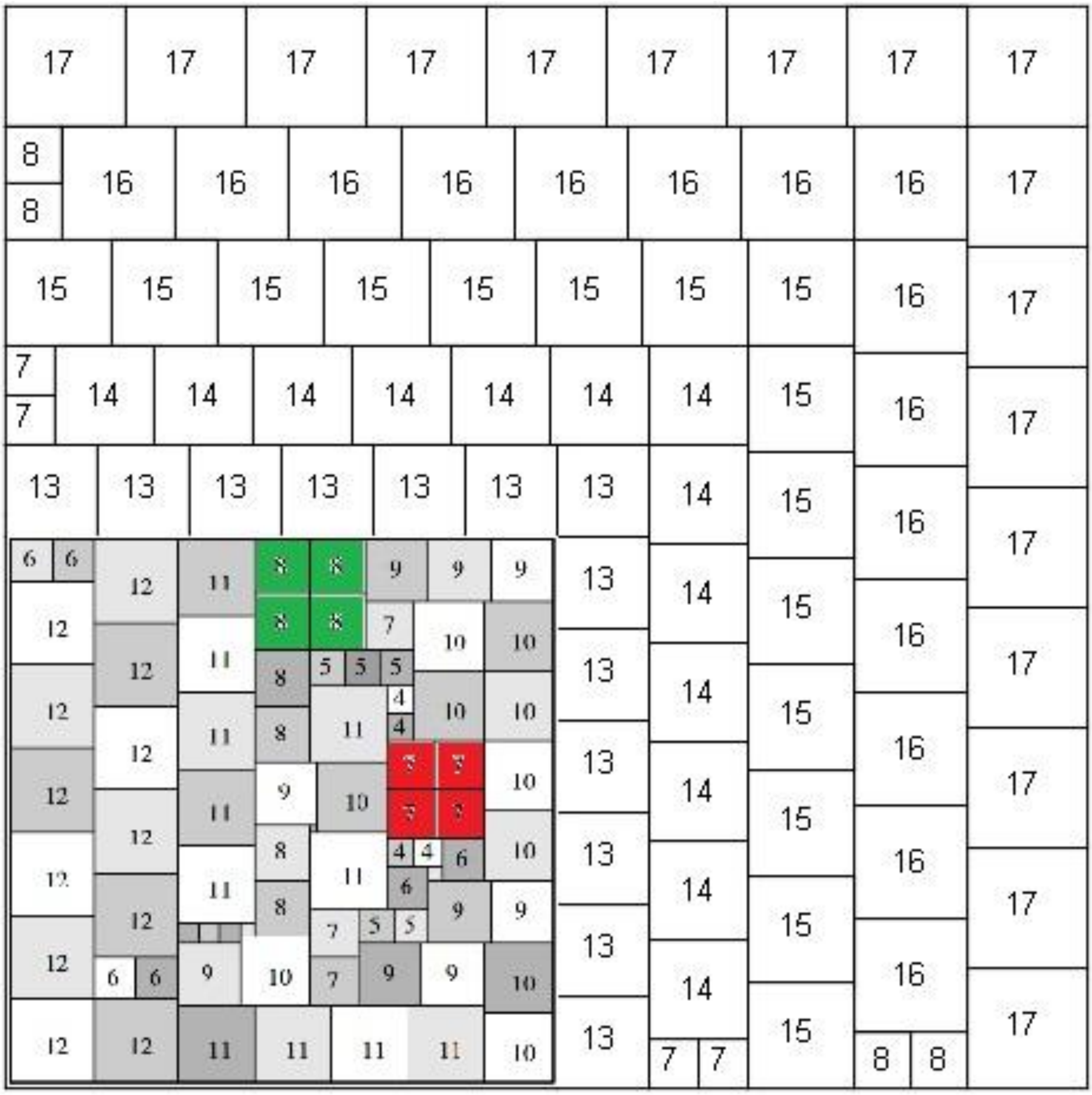
Using same solution of $36^2$ and adding the nine 9x9 squares in 2 borders

$55^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 + 7x\ 7^2 + 8x\ 8^2 + 9x\ 9^2 + 10x\ 10^2 = 0$ by Pontus von Brömssen



$66^2 = 1x\ 1^2 + 2x\ 2^2 + 3x\ 3^2 + 4x\ 4^2 + 5x\ 5^2 + 6x\ 6^2 + 7x\ 7^2 + 8x\ 8^2 + 9x\ 9^2 + 10x\ 10^2 + 11x\ 11^2 = 0$ by Pontus von Brömssen

Using same solution of $55^2$ and adding the elven 11x11 squares in 2 borders



Carl F. Schwenke comment that you can use the solution for N12=0 by Ågren et al. to construct solutions for N13=N14=N15=N16=N17=0.

## PROBLEM 2:

https://demonstrations.wolfram.com/TightlyPackedSquares/

https://oeis.org/A081287

Put squares from 1 to N in the smallest possible rectangle.

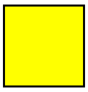
What is the smallest area that is left uncovered?

The sequence for OEIS at the moment is: 0,1,1,5,8,8,14,6,15,20

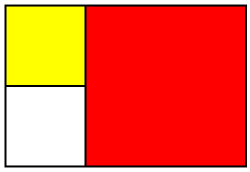
| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| A | 1 | 6 | 15 | 35 | 63 | 99 | 154 |
| L | 0 | 1 | 1 | 5 | 5 | 8 | 14 |

| N | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|
| A | 210 | 300 | 405 | | | | |
| L | 6 | 15 | 20 | | | | |

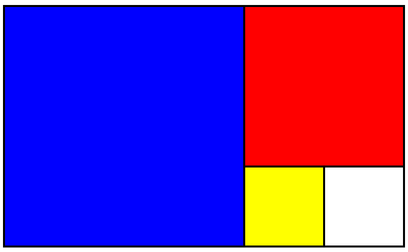
N = 1: 1x1, left = 0



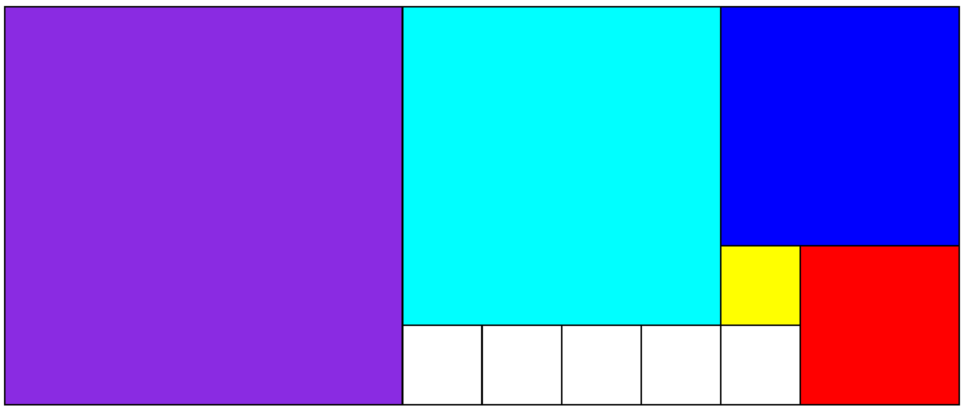
N = 2: 2x3 = 6, left = 1



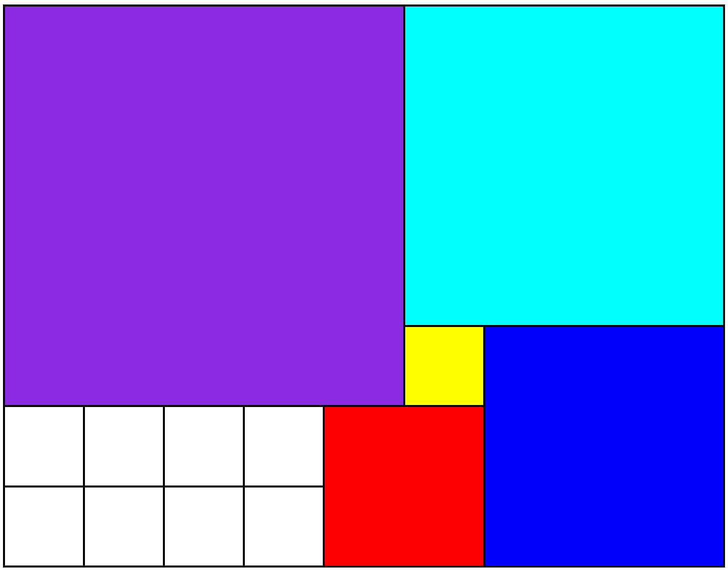
N = 3: 3x5 = 15, left = 1



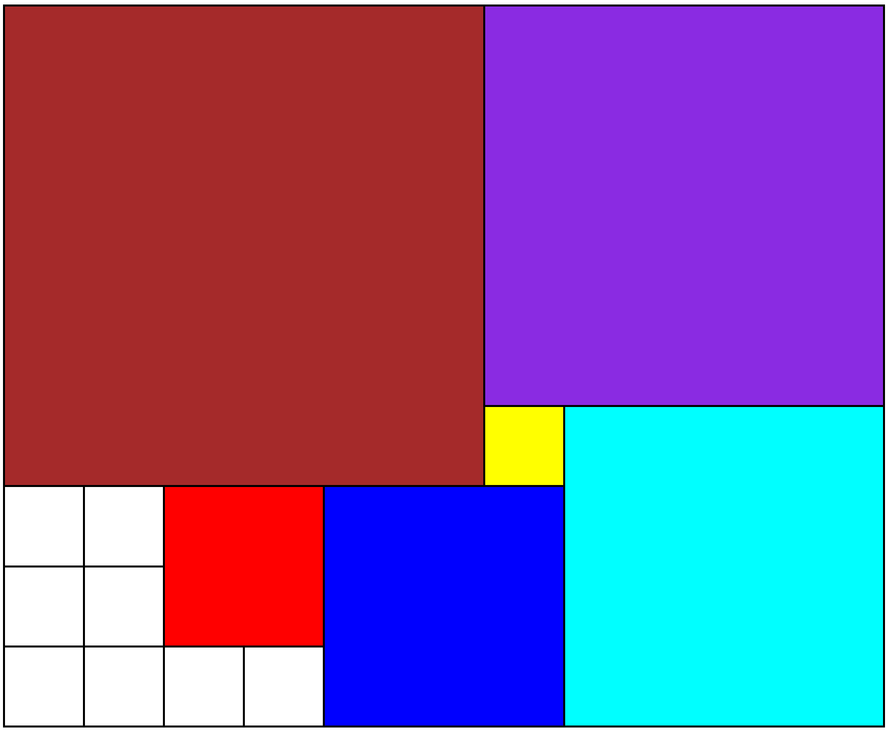
N = 4: 5x7 = 35, left = 5



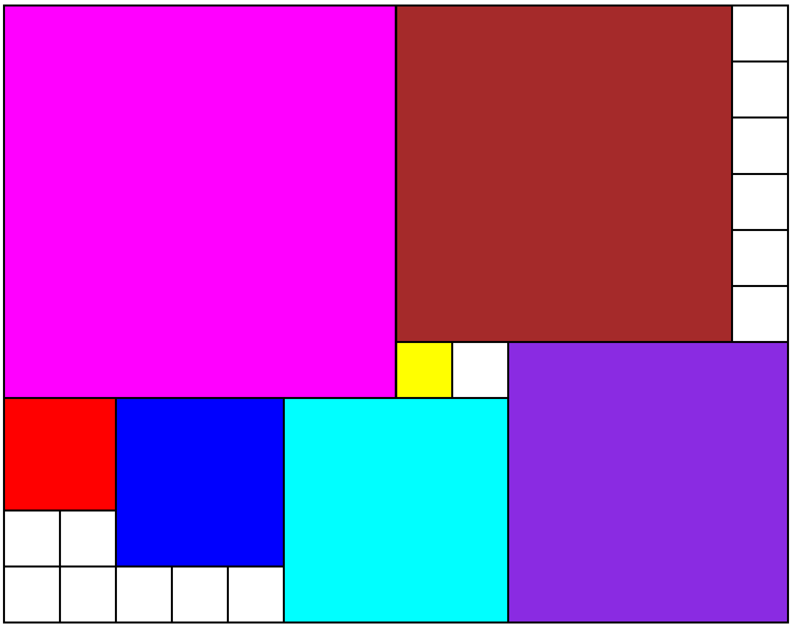
N = 5: 12x5 = 60, left = 5



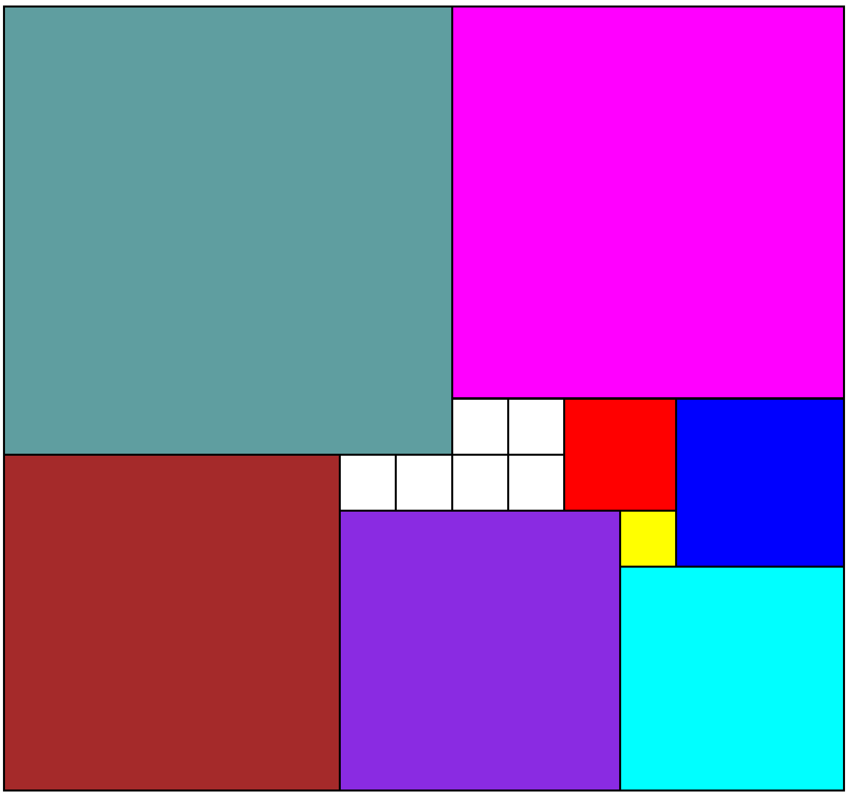
N = 6: 9x11 = 99, left = 8



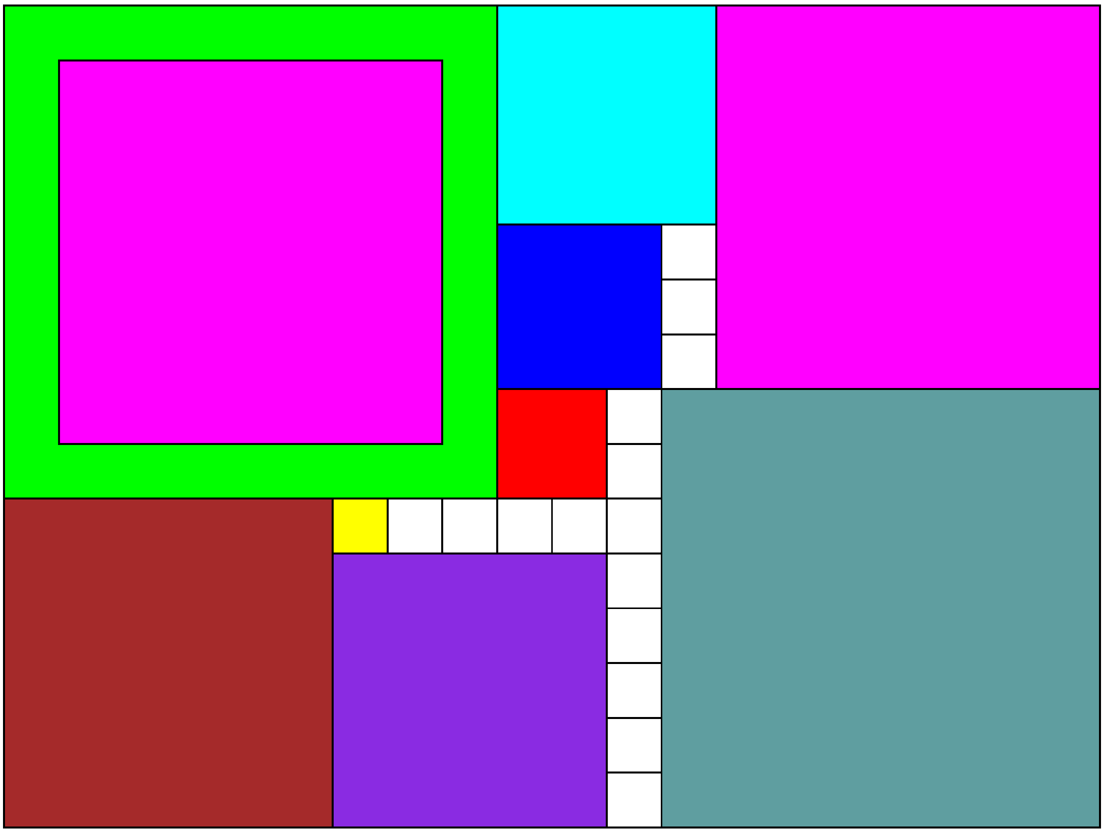
N =7: 14x11 = 154, left = 14



N =8: 15x14 = 210, left = 6



N =9: 20x15 = 300, left = 15



N =10: 27x15 = 405, left = 20